Automorphic Correction of Kac-Moody Algebras

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0.1. **Kac-Moody Algebra.** Let I be a countable set of indices. Let $A = (a_{ij})_{i,j \in I}$ be a matrix with entries in $\mathbb{R}$, satisfying the following conditions:

(a) A is symmetric and $a_{ii} > 0$, and $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}$ for all $j \in I$.

(b) $a_{ij} \leq 0$ if $i \neq j$.

(c) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Definition 0.1. The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ associated to the Cartan matrix A is defined to be the Lie algebra with generators $e_i, h_i, f_i (i \in I)$ and the following defining relations:

(i) $[h_i, h_j] = 0,$

(ii) $[h_i, e_k] = a_{ik} e_k, \quad [h_i, f_k] = -a_{ik} f_k,$

(iii) $[e_i, f_j] = \delta_{ij} h_i,$

(iv) $(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \quad$ for $i \neq j$.

Let V be a highest weight $\mathfrak{g}$-module with highest weight λ.

Then we have the Weyl-Kac character formula:

$$chV = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Phi_+} (1 - e^{-\alpha})^{m(\alpha)}},$$

where $m(\alpha)$ is the root multiplicity of α.

Letting $\lambda = 0$, we obtain the denominator identity:

$$\prod_{\alpha \in \Phi_+} (1 - e^{-\alpha})^{m(\alpha)} = \sum_{w \in W} (-1)^{l(w)} e^{w\rho - \rho}.$$

Kac said in 1997, "It is a well kept secret that the theory of Kac-Moody algebras has been a disaster."

It is due to the fact that there is no single Kac-Moody algebra beyond affine case where root multiplicity formulas are known.

On the other hand, many Borcherds algebras have explicit root multiplicities as Fourier coefficients of modular forms. Gritsenko and Nikulin introduced the notion of automorphic correction, originated in Borcherds' work. Namely, given a Kac-Moody algebra, we can embed the Kac-Moody algebra $\mathfrak{g}$ into a Borcherds superalgebra $\mathfrak{G}$ by adding positive roots, whose denominator in the denominator identity becomes a modular form.

We call $\mathfrak{G}$ automorphic correction of $\mathfrak{g}$.

0.2. **Example of Kac-Moody algebra.** Let $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$. It gives rise to a hyperbolic Kac-Moody algebra. Lepowsky and Moody showed that there is one to one correspondence between the root system and quasi-regular cusps of Hilbert modular sufraces attached to $K = \mathbb{Q}[\sqrt{5}]$.

A. Feingold showed that the root system is related to Fibonacci numbers: Let a_i be the Fibonacci number, given by $a_0 = 0, a_1 = 1$ and $a_i = a_{i-1} + a_{i-2}$ for $i \in \mathbb{Z}$.

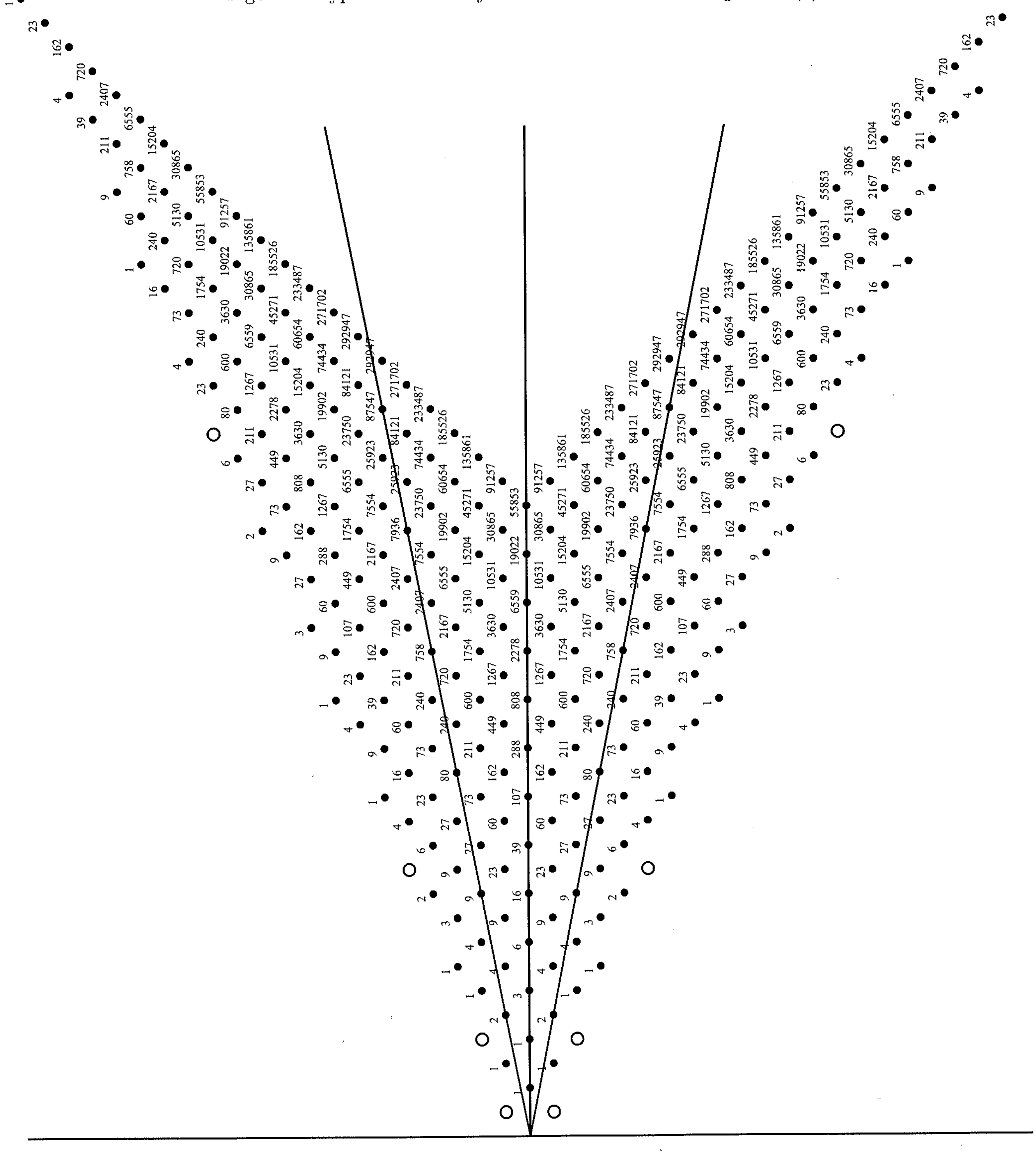
Then the denominator identity is

$$\prod_{\substack{i\in\mathbb{Z} \\ p\geq 0, q\geq 1}} (1 - u^{pa_{2i-1}+qa_{2i+1}} v^{qa_{2i-1}+pa_{2i-3}})^{C(p,q-p)} \prod_{i\geq 0} (1 - u^{a_{2i}} v^{a_{2i+2}})(1 - u^{a_{2i+2}} v^{a_{2i}})$$

$$= u^{-1} v^{-1} \sum_{j\in\mathbb{Z}} (-1)^j u^{a_{2j+1}} v^{a_{2j-1}}.$$

The root multiplicity $C(p, q-p)$ is unknown. Here is a root system and root multiplicities.

Figure 2: Hyperbolic Root System For The Fibonacci Algebra $\mathcal{H}(3)$



Let $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. It gives rise to a hyperbolic Kac-Moody algebra.

Let $P = \begin{pmatrix} 3 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}$, $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$, and $T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, and $a, b, c \in \mathbb{Z}$.

The denominator identity is

$$\prod_{T \geq 0} (1 - e^{2\pi i Tr(TZ)})^{mult(T)} \prod_{T \in \Delta_{re}} (1 - e^{2\pi i Tr(TZ)}) = \sum_{g \in PGL_2(\mathbb{Z})} det(g) e^{2\pi i Tr(gPg^t - P)Z},$$

where

$\Delta_{re} = \{T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, and $a, b, c \in \mathbb{Z}$, $ac - b^2 = -1$, $a + c \geq b$, $a + c \geq 0$, $c \geq 0\}$.

Here $mult(T)$ is unknown. For some roots, $mult(T)$ is $p(det(T) + 1)$, where $p(n)$ is the partition function.

0.3. **Borcherds-Kac Algebra.** Let I be a countable set of indices. Let $A = (a_{ij})_{i,j \in I}$ be a matrix with entries in $\mathbb{R}$, satisfying the following conditions:

(a) A is symmetric,

(b) if $i \neq j$ then $a_{ij} \leq 0$,

(c) if $a_{ii} > 0$ then $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}$ for all $j \in I$.

Definition 0.2. The *Borcherds-Kac algebra* $\mathfrak{g} = \mathfrak{g}(A)$ associated to the matrix A is defined to be the Lie algebra with generators e_i, h_i, f_i $(i \in I)$ and the following defining relations:

(i) $[h_i, h_j] = 0$,

(ii) $[h_i, e_k] = a_{ik} e_k, \quad [h_i, f_k] = -a_{ik} f_k$,

(iii) $[e_i, f_j] = \delta_{ij} h_i$,

(iv) $(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \quad$ for $i \neq j$ and $a_{ii} > 0$,

(v) $[e_i, e_j] = 0, \quad [f_i, f_j] = 0 \quad$ if $a_{ij} = 0$.

The denominator identity is given by:

$$\prod_{\alpha\in\Phi_+}(1-e^{-\alpha})^{m(\alpha)} = e(-\rho)\sum_{w\in W}(-1)^{l(w)}w(e(\rho))\sum_{\Psi}(-1)^{|\Psi|}e(-\sum\Psi)),$$

where Ψ runs over all finite subsets of mutually orthogonal imaginary fundamental roots.

Example (fake monster algebra)

Let S be a hyperbolic even unimodular lattice of signature (25,1).

Then $S = [\rho, e] \oplus L$, where L is the Leech lattice, positive definite even unimodular lattice of rank 24.

The Gram matrix of ρ, e is $H = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

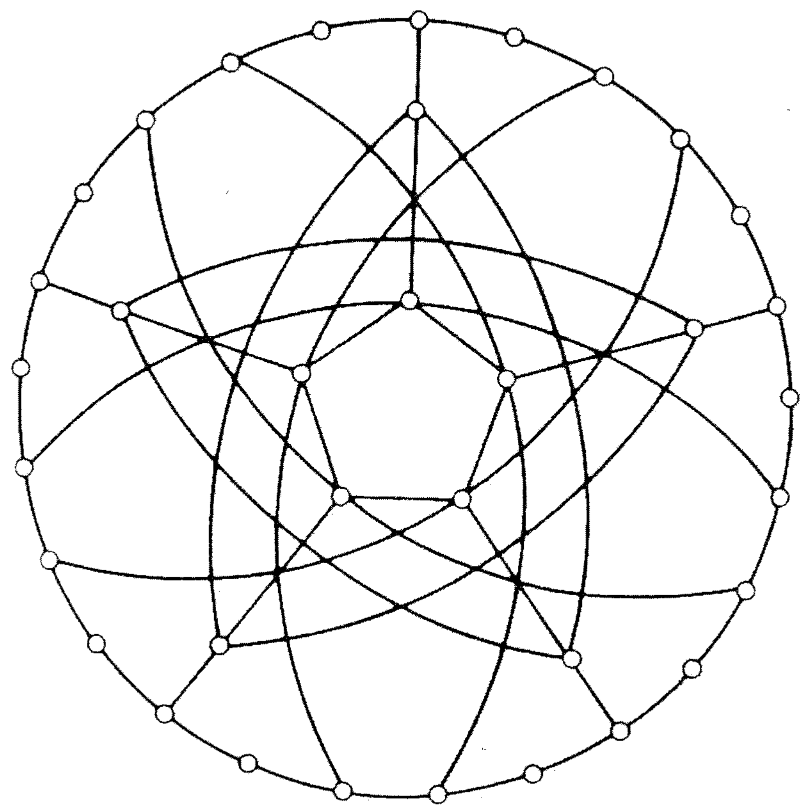
Let $P = \{\alpha \in S : (\alpha, \alpha) = 2, (\rho, \alpha) = -1\}$.

Then $A = ((\alpha, \alpha'))$, $\alpha, \alpha' \in P$, is a generalized Cartan matrix, and

A defines a Kac-Moody algebra $\mathfrak{g}(A)$. However, the denominator is not a modular form.

We add more positive roots so that $\mathfrak{g}(A) \subset \mathfrak{g}(A')$.

The following graph shows a section of the Dynkin diagram of the fake monster algebra



Let $P' = P \cup 24\rho \cup 24(2\rho) \cup \cdots \cup 24(n\rho) \cup \cdots$

Here $24(n\rho)$ means that we take $n\rho$ 24 times.

Let $A' = ((\alpha, \alpha'))$ for $\alpha, \alpha' \in P'$.

Then A' defines a Borcherds-Cartan matrix and gives rise to a Borcherds-Kac algebra $\mathfrak{g}(A')$.

The denominator identity is

$$\Phi(z) = e^{-2\pi i(\rho,z)} \prod_{\alpha\in\Delta_+} (1 - e^{-2\pi i(\alpha,z)})^{p_{24}(1-\frac{(\alpha,\alpha)}{2})} = \sum_{w\in W} det(w) \sum_{m>0} \tau(m) e^{-2\pi i(w(m\rho),z)},$$

where

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{m=1}^{\infty}\tau(m)q^m, \quad \Delta^{-1} = \sum_{n=0}^{\infty} p_{24}(n)q^{n-1}.$$

Here $\Phi(z)$ is an automorphic form of weight 12 with respect to $O^+(T)$, where $T = H \oplus S$ is an extended lattice of signature (26,2).

This is the first instance of automorphic correction.

0.4. **Borcherds-Kac superalgebra.** A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $[a,b] = -(-1)^{d(a)d(b)}[b,a]$ and $[a,[b,c]] = [[a,b],c] + (-1)^{d(a)d(b)}[[a,c],b]$.

Let I be a countable set of indices. Let $A = (a_{ij})_{i,j \in I}$ be a matrix with entries in $\mathbb{R}$, satisfying the following conditions, and let S be a subset of I.

(a) A is symmetric; if $i \neq j$ then $a_{ij} \leq 0$,

(b) if $a_{ii} > 0$, then $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}$ for all $j \in I$.

(c) if $a_{ii} > 0$ and $i \in S$, then $\frac{a_{ij}}{a_{ii}} \in \mathbb{Z}$ for all $j \in I$.

Definition 0.3. The *Borcherds-Kac superalgebra* $\mathfrak{g} = \mathfrak{g}(A)$ associated to the matrix A and S is defined to be the Lie superalgebra with generators e_i, h_i, f_i $(i \in I)$ and the following defining relations:

(i) $[h_i, h_j] = 0; \quad [h_i, e_k] = a_{ik} e_k, \quad [h_i, f_k] = -a_{ik} f_k,$

(ii) $[e_i, f_j] = \delta_{ij} h_i,$

(iii) $deg(e_i) = 0 = deg(f_i)$ if $i \notin S$, and $deg(e_i) = deg(f_i) = 1$ if $i \in S$.

(iv) $(\mathrm{ad}\, e_i)^{1-2a_{ij}/a_{ii}} e_j = 0, \quad (\mathrm{ad}\, f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \quad$ for $i \neq j$ and $a_{ii} > 0$,

(v) $(\mathrm{ad}\, e_i)^{1-a_{ij}/a_{ii}} e_j = 0, \quad (\mathrm{ad}\, f_i)^{1-a_{ij}/a_{ii}} f_j = 0 \quad$ for $i \neq j$ and $a_{ii} > 0$, and $i \in S$,

(vi) $[e_i, e_j] = 0, \quad [f_i, f_j] = 0 \quad$ if $a_{ij} = 0$.

Then we have the decomposition $\Delta_+ = \Delta_+^0 \bigcup \Delta_+^1$ such that if $\alpha \in \Delta_+^0$ if and only if $\mathfrak{g}_\alpha \subset G_{\bar{0}}$.

We have the denominator identity:

$$\prod_{\alpha \in \Delta_+^0} (1 - e(-\alpha))^{m_0(\alpha)} \prod_{\alpha \in \Delta_+^1} (1 - e(-\alpha))^{-m_1(\alpha)} = e(-\rho) \sum_{w \in W} det(w) w(T),$$

where $T = e(\rho) \sum_\mu (-1)^{ht_0(\mu)} e(-\mu)$.

Here if $\mu = \sum_{i \in I} k_i \alpha_i$, then $ht_0(\mu) = \sum_{i \in I - S} k_i$.

0.5. **Automorphic Correction.** We recall the theory of automorphic correction established by Gritsenko and Nikulin. The original idea of automorphic correction can be traced back to Borcherds' work. We assume that the following data (1)-(4) are given.

(1) We are given a lattice M with a non-degenerate integral symmetric bilinear form $(\cdot,\cdot)$ of signature $(n,1)$ for some $n \in \mathbb{N}$.
(2) A nontrivial reflection group $W \subset O(M)$ is given. The group W is generated by reflections in some roots of the lattice M. A vector $\alpha \in M$ is called a root if $(\alpha,\alpha) > 0$ and (α,α) divides $2(\alpha,\beta)$ for all $\beta \in M$.
(3) Consider the cone

$$V(M) = \{\beta \in M \otimes \mathbb{R} \,|\, (\beta,\beta) < 0\},$$

which is a union of two half cones. One of these half cones is denoted by $V^+(M)$. Choose a minimal set Π of roots orthogonal to a fundamental chamber $\mathcal{M} \subset V^+(M)$ of W so that $\mathcal{M} = \{\beta \in V^+(M) \,|\, (\beta,\alpha) \le 0 \text{ for all } \alpha \in \Pi\}$.

Moreover, we have a Weyl vector $\rho \in M \otimes \mathbb{Q}$ satisfying

$(\rho, \alpha) = -(\alpha, \alpha)/2$ for each $\alpha \in \Pi$.

(4) Define the complexified cone $\Omega(V^+(M)) = M \otimes \mathbb{R} + iV^+(M)$.

Let $L = \begin{pmatrix} 0 & -m \\ -m & 0 \end{pmatrix} \oplus M$ be an extended lattice for some $m \in \mathbb{N}$.

We consider the quadratic space $V = L \otimes \mathbb{Q}$ and obtain $\mathcal{K}^+$.

Define a map $\Omega(V^+(M)) \to \mathcal{K}$ by $z \mapsto \left[\frac{(z,z)}{2m} e_1 + e_2 + z\right]$,

where $\{e_1, e_2\}$ is the basis for $\begin{pmatrix} 0 & -m \\ -m & 0 \end{pmatrix}$.

Then the space $\mathcal{K}^+$ is canonically identified with $\Omega(V^+(M))$.

We are given a holomorphic automorphic form $\Phi(z)$ on $\Omega(V^+(M))$ with respect to a subgroup $\Gamma \subset O_L^+$ of finite index.

The automorphic form Φ has a Fourier expansion of the form

$$\Phi(z) = \sum_{w \in W} \det(w) \left(e\left(-(w(\rho), z)\right) - \sum_{a \in M \cap \mathcal{M}} m(a)\, e(-(w(\rho + a), z)) \right),$$

where $e(x) = e^{2\pi i x}$ and $m(a) \in \mathbb{Z}$ for all $a \in M \cap \mathcal{M}$.

The matrix

$$A = \left(\frac{2(\alpha,\alpha')}{(\alpha,\alpha)}\right)_{\alpha,\alpha'\in\Pi}$$

defines a Kac-Moody algebra $\mathfrak{g}$. Moreover, the data (1)-(4) define a Borcherds-Kac superalgebra $\mathcal{G}$.

We call $\mathcal{G}$ (or $\Phi(z)$) an *automorphic correction* of $\mathfrak{g}$. The automorphic form $\Phi(z)$ determines the set of simple imaginary roots of $\mathcal{G}$, and can be written, using the denominator identity for the Borcherds-Kac superalgebra $\mathcal{G}$, as the product

$$\Phi(z) = e(-(\rho, z)) \prod_{\alpha\in\Delta(\mathcal{G})^+} (1 - e(-(\alpha, z)))^{\mathrm{mult}(\mathcal{G},\alpha)},$$

where $\Delta(\mathcal{G})^+$ is the set of positive roots of $\mathcal{G}$ and $\mathrm{mult}(\mathcal{G}, \alpha)$ is the root multiplicity of α in $\mathcal{G}$.

Example: The automorphic correction of the Kac-Moody algebra $\mathfrak{g}(A)$ attached to

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

is the Siegel cusp form of weight 35 (called Igusa modular form):

Let $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$, and $q = e^{2\pi i z_1}, r = e^{2\pi i z_2}, s = e^{2\pi i z_3}$.

$$\Delta_{35}(Z) = q^2 r s^2 (q - s) \prod_{\substack{n,l,m \in \mathbb{Z} \\ (n,l,m)>0}} (1 - q^n r^l s^m)^{f_2(4nm - l^2)},$$

where $f_2(4nm - l^2)$ is defined by

$$f_2(N) = 8f(4N) + 2((\frac{-N}{2}) - 1)f(N) + f(\frac{N}{4}),$$

and $f(N) = \begin{cases} f(n,l), & \text{if } N = 4n - l^2 \\ 0, & \text{otherwise} \end{cases}$, and $(\frac{D}{2}) = \begin{cases} 1, & \text{if } D \equiv 1 \pmod 8 \\ -1, & \text{if } D \equiv 5 \pmod 8 \\ 0, & \text{if } D \equiv 0 \pmod 2 \end{cases}$.

Here $f(n,l)$ is the Fourier coefficient of a weak Jacobi form of weight 0 and index 1:

$$\phi_{0,1}(z_1,z_2) = \phi_{12,1}(z_1,z_2)/\Delta_{12}(z_1) = \sum_{n\geq 0, l\in\mathbb{Z}} f(n,l)e^{2\pi i(nz_1+lz_2)}.$$

Here the reason why Siegel modular forms enter is because of the identification $O(3,2) \sim Sp_4$.

Also note that the Jacobi form of weight k and index 1 is canonically isomorphic to half-integral weight modular forms of weight $\frac{k-1}{2}$.

Borcherds showed that modular forms of weight $-\frac{n-2}{2}$ give rise to modular forms on $O(n,2)$.

Now we have identification $O(2,2) \sim SL_2 \times SL_2$.

Hence modular forms on $O(2,2)$ are Hilbert modular forms.

It makes sense to find automorphic corrections of

the Kac-Moody algebra attached to $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ among Hilbert modular forms.

Positive roots are subset of $\mathcal{O}_K = \mathbb{Z}[\epsilon_0]$, where $\epsilon_0 = \frac{1+\sqrt{5}}{2}$. Let $\eta = \frac{3+\sqrt{5}}{2} = \epsilon_0^2$.

We will use the column vector notation for the elements in $\mathfrak{h}^*$

with respect to the basis $\{\gamma^+, \gamma^-\}$, i.e. we write $\binom{x}{y}$ for $x\gamma^+ + y\gamma^-$,

where $\gamma^+ = \alpha_1 + \bar{\eta}\alpha_2$, $\gamma_- = \alpha_1 + \eta\alpha_2$.

Then we have

$$\alpha_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta \\ -\bar{\eta} \end{pmatrix} \quad \text{and} \quad \alpha_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

It is now easy to see that $\mathfrak{h}_{\mathbb{Q}}^* = \{\binom{x}{\bar{x}} \mid x \in F\}$.

We define a map $\psi : \mathfrak{h}_{\mathbb{Q}}^* \to F$ by $\binom{x}{\bar{x}} \mapsto x$.

Then the map ψ is an isometry from $(\mathfrak{h}_{\mathbb{Q}}^*, (\cdot,\cdot))$ to $(F, \langle\cdot,\cdot\rangle)$.

A symmetric bilinear form $\langle\cdot,\cdot\rangle$ on F is defined by $\langle x, y\rangle = -5\,\mathrm{tr}(xy')$.

In particular, the root lattice $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is mapped onto

a sublattice of $\mathcal{O}_K/\sqrt{5}$.

We denote the simple reflection corresponding to α_i by r_i.

The Weyl group W acts on K as: $r_1 x = \eta^2 \bar{x}$ and $r_2 x = \bar{x}$. So

$$r_1 = \begin{pmatrix} 0 & \eta^2 \\ \bar{\eta}^2 & 0 \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and

$$W = \{(r_1 r_2)^i, r_2(r_1 r_2)^i \,|\, i \in \mathbb{Z}\}.$$

Then we have

$$\psi(\Delta_{re}^+) = \{\frac{\eta^i}{\sqrt{5}}, i > 0, \quad -\frac{\bar{\eta}^i}{\sqrt{5}}, i \geq 0\},$$

$$\psi(\Delta_{\text{im}}^+) = \left\{ \frac{1}{\sqrt{p}}\eta^j(m\eta - n),\ \frac{1}{\sqrt{p}}\eta^j(n\eta - m),\ \frac{1}{\sqrt{p}}\bar{\eta}^j(n - m\bar{\eta}),\ \frac{1}{\sqrt{p}}\bar{\eta}^j(m - n\bar{\eta}) \right\},$$

where $j \geq 0$ and $(m, n) \in \Omega_k$ for $k \geq 1$, where

$$\Omega_k = \left\{ (m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : \sqrt{\frac{4k}{a^2 - 4}} \leq m \leq \sqrt{\frac{k}{a - 2}},\ n = \frac{am - \sqrt{(a^2 - 4)m^2 - 4k}}{2} \right\}$$

for $k \geq 1$.

Weight 0 modular form with respect to $\Gamma_0(5)$:

$$f(z) = \frac{E_2^+(z)}{H^{(q)}(z)} = q^{-1} + 5 + 11q - 54q^4 + 55q^5 + \cdots = q^{-1} + \sum_{n=0}^{\infty} a(n)q^n,$$

where

$$E_2^+(z) = 1 - 5\sum_{n=1}^{\infty}\sum_{d|n} d(\chi_5(d) + \chi_5(\frac{n}{d}))q^n, \quad H^{(q)}(z) = \frac{\eta(5z)^5}{\eta(z)}.$$

Borcherds lift of $f(z)$ is the Hilbert modular form of weight 5:

$$\Psi_1(z_1, z_2) = e(\frac{\epsilon_0 z_1}{\sqrt{5}} - \frac{\epsilon_0' z_2}{\sqrt{5}}) \prod_{\substack{\nu \in \frac{1}{\sqrt{5}}\mathcal{O}_K \\ \epsilon_0 \nu' - \epsilon' \nu > 0}} (1 - e(\nu z_1 + \nu' z_2))^{s(5\nu\nu')a(5\nu\nu')},$$

where ν' is the conjugate of ν in $K = \mathbb{Q}[\sqrt{5}]$ and $s(n) = \begin{cases} 2, & \text{if } 5|n \\ 1, & \text{otherwise} \end{cases}.$

Let $\Phi_1(z_1, z_2) = \overline{\Psi_1}(5z_1, 5z_2)$.

Then Ψ_1 is a Hilbert modular form with respect to $\Gamma_0(5)$, and provides the automorphic correction for the Kac-Moody algebra $\mathfrak{g}(A)$ for $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$.

In particular, there exists a Borcherds-Kac superalgebra $\mathfrak{G}$ whose denominator function is the Hilbert modular form Φ_1, and $\mathfrak{g}(A) \hookrightarrow \mathfrak{G}$.